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Effect of dimensionality on the real-space renormalisation group

Shunichi Muto†, Takehiko Oguchi and Ikuo Ono

Department of Physics, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo 152, Japan

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Abstract. The first-order cumulant expansion of the real-space renormalisation group is applied to the Ising model on d-dimensional hypercubic lattices in $d = 2, 3, 4$ and 5. These calculations enable us to study the effect of dimensionality on the first-order cumulant expansion. Monte Carlo methods are used to make the calculations possible. One result is that the scaling power a_H shows the effect of the critical dimension, while the scaling power a_ϵ does not.

1. Introduction

There has been much work using the real-space renormalisation group (RSRG) since Niemeijer and Van Leeuwen (1973) introduced the formalism. With a few exceptions (Hilhorst *et al* (1978), and various exact calculations in $d = 1$), however, the RSRG approaches involve approximations such as the cluster expansion, the cumulant expansion and the variational method whose natures have not yet been well understood. If we could treat the RSRG rigorously in all dimensions, we should expect its results to coincide with those of high-temperature expansions and to reveal the mean field exponents in dimensions higher than or equal to the critical dimension of 4. However, as long as the RSRG remains an approximation, the question arises what the effect of dimensionality on the RSRG is. As an attempt to answer this question (because the result naturally depends on the approximation used), we have calculated the critical exponents of the Ising model on the d-dimensional hypercubic lattice, using the simplest approximation. This approximation is the first-order cumulant expansion (1OC) where we choose a cell as the d-dimensional hypercube containing 3^d sites (figure 1). Since it is virtually impossible to perform calculations by hand in dimensions higher than 4, we used the Monte Carlo (MC) method (Ma 1976, Friedman and Felsteiner 1977, Rácz and Ruján 1977) to evaluate the 1OC. Furthermore, as a check on the accuracy of the MC, we have calculated the 1OC for $d = 2$ and 3, where we can compare with results obtained without using the MC method (Hsu *et al* 1975, Hsu and Gunton 1977).

† Present address: Department of Physics, Boston University, 111 Cummington Street, Boston, MA 02215, USA

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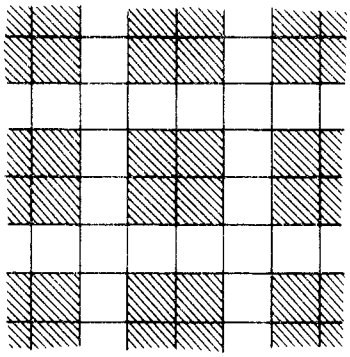

Figure 1. Cells ($d = 2$). The shaded areas are the zeroth-order cells. The interactions between cells (the unshaded area) are treated as a perturbation.

2. Formulae used in the MC calculation

The recursion relation for the nearest-neighbour interaction K in the case of 1OC is (Niemeijer and Van Leeuwen 1973, Rácz and Ruján 1977)

$$K' = Kf(K) \tag{1}$$

where

$$f(K) = \sum_n{}' \langle S_i^n \operatorname{sgn} M \rangle^2. \tag{2}$$

Here S_i^n is the Ising spin on the nth site of the ith cell, and $M = \Sigma_m S_i^m$. (We do not have to consider more than one cell in the recursion relations of the 1OC.) The Σ_n' indicates that a summation is taken over the 3^{d-1} sites on one surface of the cell. The bracket $\langle \cdots \rangle$ denotes the usual thermal average within the cell.

The critical temperature $T_c (= J/k_B K^*)$ is determined by the fixed point condition (cf equation (1))

$$f(K^*) = 1. \tag{3}$$

The thermal eigenvalue of the linearised recursion relation is

$$\lambda_T = \left(\frac{dK'}{dK}\right)_{K=K^*} = 1 + K^* f'(K^*). \tag{4}$$

To calculate $f'(K^*)$ by the MC method, we have two choices: (i) numerical differentiation and (ii) a fluctuation calculation. We have adopted the latter. Hence:

$$\lambda_T = 1 + 2K^* \sum_n{}' \langle S_i^n \operatorname{sgn} M \rangle \left(\langle S_i^n E \operatorname{sgn} M \rangle - \langle S_i^n \operatorname{sgn} M \rangle \langle E \rangle \right). \tag{5}$$

Here $E = \Sigma_{\langle kl \rangle} S_i^k S_i^l$, and the summation $\Sigma_{\langle kl \rangle}$ is taken over all nearest-neighbour spin pairs in the cell.

The magnetic eigenvalue associated with the odd part (see Niemeijer and Van Leeuwen 1973) of the linearised recursion relation is

$$\lambda_H = \langle |M| \rangle + 2dK^* \sum_n{}' \langle S_i^n \operatorname{sgn} M \rangle \left(\langle S_i^n M \rangle - \langle |M| \rangle \langle S_i^n \operatorname{sgn} M \rangle \right). \tag{6}$$

We determine the numerical value K^* by a least square fit of the numerical results for $f(K)$ at 21 different K values. The data for each K value are obtained using 1000 MC steps/spin. The eigenvalues λ_T and λ_H are determined in a similar fashion.

3. Results and discussion

Since MC is a statistical method, the results differ from run to run. Therefore we calculated the mean values and the variances (errors) from 4 independent MC runs. One MC run corresponds to 21×1000 MC steps/spin. We present our results in table 1 and figure 2. Note that our results in table 1 are consistent with the $d = 2, 3$ results of Hsu and Gunton (1977).

Table 1. Results. The errors indicated are estimates due to 4 MC runs. [a], Hsu and Gunton (1977); *,**, Stanley 1971. $a_\epsilon^{-1} = d \ln 3/\ln \lambda_T$, $\alpha = 2 - a_\epsilon^{-1}$; $a_H^{-1} = d \ln 3/\ln \lambda_H$, $1/\delta = a_H^{-1} - 1$.

d	K_c	a_ϵ^{-1}*	α	a_H^{-1}**	δ
2	$0{\cdot}473 \pm 0{\cdot}005$	$2{\cdot}142 \pm 0{\cdot}03$	$-0{\cdot}142 \pm 0{\cdot}03$	$1{\cdot}028 \pm 0{\cdot}003$	$35{\cdot}7 \pm 4$
	($0{\cdot}4697^a$)	($2{\cdot}157^a$)	($-0{\cdot}157^a$)	($1{\cdot}0293^a$)	($34{\cdot}13^a$)
3	$0{\cdot}258 \pm 0{\cdot}003$	$2{\cdot}57 \pm 0{\cdot}02$	$-0{\cdot}57 \pm 0{\cdot}02$	$1{\cdot}187 \pm 0{\cdot}003$	$5{\cdot}35 \pm 0{\cdot}1$
	($0{\cdot}2599^a$)	($2{\cdot}575^a$)	($-0{\cdot}575^a$)	($1{\cdot}1853^a$)	($5{\cdot}397^a$)
4	$0{\cdot}179 \pm 0{\cdot}002$	$2{\cdot}80 \pm 0{\cdot}02$	$-0{\cdot}80 \pm 0{\cdot}02$	$1{\cdot}283 \pm 0{\cdot}003$	$3{\cdot}53 \pm 0{\cdot}03$
5	$0{\cdot}136 \pm 0{\cdot}003$	$2{\cdot}95 \pm 0{\cdot}11$	$-0{\cdot}95 \pm 0{\cdot}11$	$1{\cdot}377 \pm 0{\cdot}02$	$2{\cdot}65 \pm 0{\cdot}3$

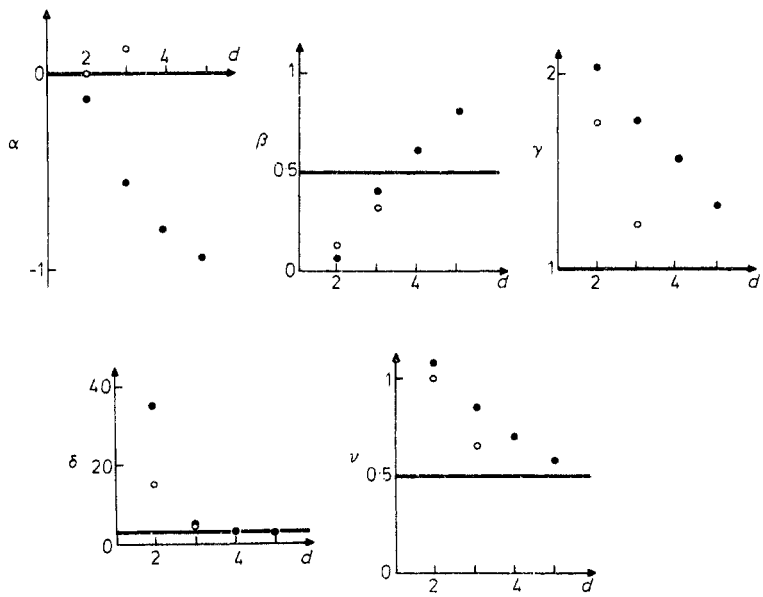

Figure 2. Exponents (α, β, γ, δ and ν) versus d. ●, present results calculated by using the data in table 1; ○, rigorous or high-temperature expansion results (see Stanley 1971); ——, molecular field results. $\alpha = 2 - a_\epsilon^{-1}$; $\beta = (1 - a_H)a_\epsilon^{-1}$; $\gamma = (2a_H - 1)a_\epsilon^{-1}$; $\delta = a_H(1 - a_H)^{-1}$; $\nu = 1/da_\epsilon$.

In $d=4$, the magnetic exponent δ is fairly near the mean field value of 3, but the exponent α is far from the mean field value of zero. Formally, however, it is better to state the accuracy of the approximation in terms of a_ϵ and a_H (Stanley 1971) because $\alpha=0$ in $d\geqslant 4$. Thus the deviation of a_H^{-1} from the exact value ($=\frac{4}{3}$) is ~4% though that of a_ϵ^{-1} from the exact value ($=2$) is ~40%. One reason for this difference might be as follows. The expansion of λ_H starts from the zeroth-order term, i.e. $\langle|M|\rangle$, while that of λ_T starts from first-order terms in the perturbation series. Consequently, the contribution of neglected second- and higher-order terms is larger in λ_T than in λ_H.

A similar difference in accuracy between the thermal and magnetic scaling powers can also be found in $d=5$. In $d>4$, however, we should expect the presence of the 'dangerous irrelevant variable' (Fisher 1973, Knops *et al* 1977) which would break some of the relations between exponents and eigenvalues such as $\nu=\ln 3/\ln \lambda_T$ and $\alpha=2-d\ln 3/\ln\lambda_T$. This would in turn cause the breakdown of hyperscaling such as $d\nu=2-\alpha$. This problem must be left for a future work since the present method (1OC) is inadequate to address it.

In summary, we have calculated the effect of the dimensionality on the first-order cumulant expansion of the RSRG and obtained the result which indicates that the magnetic exponent δ reflects the critical dimension better than the thermal exponent α does.

This is only a first step, and various extensions are possible. It is easy, for example, to enlarge the linear dimension of the cell, which was fixed at 3 here, to be $5, 7, \ldots, (2n+1)$. Only considerations of computer time limit us. Another direction is to apply other approximations such as the second-order cumulant expansion and cluster expansions. Also of great interest to us is to extend this approach to random Ising models. Recently, applications of RSRG to this problem have begun to appear (Tatsumi and Kawasaki 1976, Kinzel and Fischer 1978).

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